Proofs of Claims Leading to the Intermediate Value Theorem

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This supplement contains proofs of theorems stated in the article "The Intermediate Value Theorem Is *NOT* Obvious—and I Am Going to Prove It to You," *College Mathematics Journal* **42** (2011) 254–259, doi:10.4169/college.math.j.42.4.254. We use the sequence version of the COMPLETENESS AXIOM and the definition of *real number* as stated in that paper, and we also invoke propositions proved early on in that paper.

The proofs are presented here with an eye toward their use in a Calculus One classroom. We aim to introduce students to a mathematically correct idea of *real number* and also get them to practice thinking mathematically rather than simply adopting propositions because they "feel right." On the other hand, while we do want to expose students to a mild level of rigor, we *don't* want to overwhelm them with it. We allow as given all of the usual facts about the natural numbers (including at least an intuitive understanding of the WELL-ORDERING PRINCIPLE) and the rationals, saving the rigor for the truly new idea, that of *real number*. And to further avoid belaboring the point, we won't worry about details such as just *when* two rational sequences yield the *same* real number, or bother to verify basic properties of reals such as commutativity, associativity, and trichotomy. Those issues can still be examined outside of class time if there are questions from exceptionally curious students.

Proposition. *Every sequence of REAL numbers that has a [REAL] ceiling must have a* PERFECT *ceiling.*

Proof: Each real number in the given sequence is, by definition, the perfect ceiling of a sequence of *rational* numbers. Let's call these **short sequences**:

0:	0	0	0	0	0	...
$3/4$:	3/4	3/4	3/4	3/4	3/4	...
$\sqrt{2}$:	1	1.4	1.41	1.414	1.4142	...
Φ:	1	3/2	8/5	21/13	55/34	...
e:	1	2	5/2	16/6	65/24	...
π:	$\frac{8}{3}$	$\frac{304}{105}$	$\frac{10312}{3465}$	$\frac{135904}{45045}$	$\frac{44257352}{14549535}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We dovetail these short sequences to create a new sequence of rationals, which we'll call the **long sequence**:

0:	0	0	0	0	0	...
$3/4$:	3/4	3/4	3/4	3/4	3/4	...
$\sqrt{2}$:	1	1.4	1.41	1.414	1.4142	...
Φ:	1	3/2	8/5	21/13	55/34	...
e:	1	2	5/2	16/6	65/24	...
π:	$\frac{8}{3}$	$\frac{304}{105}$	$\frac{10312}{3465}$	$\frac{135904}{45045}$	$\frac{44257352}{14549535}$	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The long sequence has a real ceiling and hence a rational ceiling (by the earlier proposition that *for every positive real number, there is a rational number that* *is bigger*), so by the COMPLETENESS AXIOM, the long sequence has a perfect ceiling s. Then (we CLAIM) s turns out also to be the perfect ceiling for the original sequence of *real* numbers.

It seems wisest to stop here in class. We're not truly done until we prove the CLAIM just stated, of course; but by now, students have already seen the unfurling of the sequence of reals, the dovetailing of the resulting rational sequences, the application of the COMPLETENESS AXIOM to find s, and the *statement* of the CLAIM. The point is to have students think about *why* the proposition is correct—not to mention reviewing the definition of *real number*—and this has already been accomplished. It would overload most students to see the proof of the claim right away.

This is not to say that we can't come back to it later or answer questions after class, of course, so here goes:

- *The number s is a ceiling for the original sequence of reals.* Because s is a ceiling for the long sequence, it is also a ceiling for each of the short sequences, and hence it is at least as great as the *perfect* ceiling of each of the short sequences. In other words, s is at least as great as each of the reals in the original sequence.

- *The number s is the* PERFECT *ceiling for the original sequence of reals.* Any number *less than s* must fail to be a ceiling for the long sequence of rationals, hence must be less than at least one number in the long sequence; thus it must be less than that same number in one of the short sequences and, therefore, cannot be a ceiling for that particular short sequence.

  And since such a number cannot be a ceiling for that short sequence, it must be less than the PERFECT ceiling for that short sequence—that is, it must be less than the real number spawned by that short sequence of rationals.

  To sum up: Any number *less than s* must be less than some number in the original sequence of reals and, thus, must *fail* to be a ceiling for that original sequence. So s itself is the smallest ceiling, i.e. the perfect ceiling, for that original sequence.

THE CAPTURE THEOREM. *If s is the perfect ceiling* or *the perfect floor of a sequence, then any open interval containing s also contains some element of the sequence.*

THE FLIPPING-HUGE THEOREM. *If a sequence of positive real numbers has 0 as its perfect floor, then there is no ceiling for the* reciprocals *of the terms of the sequence; and conversely.*

Proof: Because it was suggested in the article that these be assigned as homework, we'll refrain from posting the proofs on the Internet!

PROPOSITION. *The "doubling sequence" 2, 4, 8, ... has no ceiling.*

Proof: First, we note that the doubling sequence cannot have any *natural-number* ceiling. For suppose it did. (This is where we use the WELL-ORDERING PRINCIPLE, though we don't need to tell students that.) Let n be the smallest natural-number ceiling for the doubling sequence. (Clearly $n > 2$.) Then $n-1$ would *not* be a ceiling for the doubling sequence and, hence, would be less than some number 2^k from the sequence. Since $n-1 < 2^k$, it would follow that $2(n-1) < 2^{k+1}$. But $n < 2(n-1)$ since $n > 2$, so we would have $n < 2^{k+1}$, contradicting the choice of n.

Next, we note that the doubling sequence cannot have any *rational* ceiling. For suppose it did have some rational ceiling $\dfrac{m}{n}$ (where m and n were both natural numbers); then

$$m = n \cdot \left(\frac{m}{n}\right) \geq \frac{m}{n},$$

so that m would be a natural-number ceiling for the sequence—contradicting what we just proved.

Finally, the doubling sequence cannot have any *real* ceiling at all. For suppose it did; then, by the earlier result that *for every positive real number, there is a rational number that is bigger*, the doubling sequence would have a rational ceiling, contradicting what we just proved.

COROLLARY: TO HALVE, AND HAVE NAUGHT. *The perfect floor of the sequence* $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ *is 0. In fact, if d is ANY positive real number, then the perfect floor of the sequence* $\frac{d}{2}, \frac{d}{4}, \frac{d}{8}, \ldots$ *is 0.*

Proof: For the first part of the corollary, because the sequence $2, 4, 8, \ldots$ has no ceiling, the sequence of reciprocals $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ has 0 for a perfect floor by the FLIPPING-HUGE THEOREM.

For the second part, if there were a positive perfect floor "t" for the sequence $\frac{d}{2}, \frac{d}{4}, \frac{d}{8}, \ldots$, then the positive real number $\frac{t}{d}$ would be a floor for the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, which would contradict the first part.

The aura theorem. *Suppose that f is* continuous *at s. If $f(s)$ is positive, then there is an open interval containing s such that f is positive on the entire interval. If $f(s)$ is negative, then there is an open interval containing s such that f is negative on the entire interval.*

Proof: Briefly, let ε be $|f(s)|/2$ and use continuity of f at s.

More precisely, because f is continuous at s, there is a $\delta > 0$ such that

$$\text{for every } x, \text{ IF } |x - s| < \delta, \text{ THEN } |f(x) - f(s)| < \varepsilon,$$

that is,

$$\text{for every } x \text{ in } (s - \delta, s + \delta), \text{ we have } f(s) - \frac{|f(s)|}{2} < f(x) < f(s) + \frac{|f(s)|}{2}.$$

It is easy to verify (and makes a good exercise to review the definition of absolute value) that these bounds on $f(x)$ both have the same sign as $f(s)$; thus, so does $f(x)$ for every x in $(s - \delta, s + \delta)$.

The intermediate value theorem (clarified). *The mathematical definition of* continuity *captures an important aspect of the informal concept of* continuity, *to wit, if f is* continuous *on $[a, b]$, and p is any number such that $f(a) < p < f(b)$ or $f(b) < p < f(a)$, then there is some c in the interval (a, b) such that $f(c) = p$.*

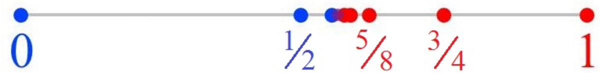
Proof: The underlying argument is as given in Simpson's reverse mathematics text [**3**, p. 87] and Mansfield's real analysis text [**1**, pp. 109–111] and presumably a dozen other sources. Though it is the kind of argument students rarely see outside a real analysis course, a dash of showmanship makes it appropriate for the calculus classroom—especially if we add a splash of color:

Class, I'm thinking of a function f that is continuous *on the closed interval $[0, 1]$. I won't show you the graph of the function, but I will tell you that it's positive at $x = 0$ and negative at $x = 1$. I'll represent this on the x-axis by coloring* 0 cyan *(for "positive") and* 1 red *(for "negative"):*

Now, do you believe that the function f ever takes the value 0 *on this interval?... You do? Then let's play a game. You, as a class, will guess an x where you think $f(x)$ might be equal to* 0*. If you're right, you win. If you're wrong, I color that x cyan or red, and you guess again. And so on.*

The GOOD *news is that you may have as many guesses as you want. The* BAD *news is that I don't guarantee the function EVER takes the value 0, so you may be guessing forever. Have at it!*

Students hit upon the most natural approach almost immediately: Guess $\frac{1}{2}$ first, then $\frac{1}{4}$ or $\frac{3}{4}$ depending on what color $\frac{1}{2}$ was, and so on:



Three things can happen:

1. There is a greatest *cyan* number, but there is no smallest *red* number.
2. There is a smallest *red* number, but there is no greatest *cyan* number.
3. There is neither a greatest cyan number nor a smallest red number.

It is worthwhile to illustrate the cases and ask the students to compare, for each case, the perfect ceiling of the cyans and the perfect floor of the reds. *They seem always to be the same.* Can we *prove* such a claim?

> *Proof of the claim.* First, the perfect ceiling of the cyans (call it "c") cannot be *greater* than the perfect floor of the reds (call it "r"). If it were, then we could look in the open interval (r, c) and find, in its right half, at least one cyan number (otherwise c would not be the *perfect* ceiling of the cyans) and, in its left half, at least one red number (otherwise r would not be the *perfect* floor of the reds).
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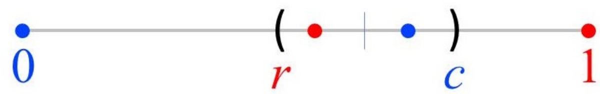
> 
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>
> Then that particular cyan would be greater than that particular red, contrary to how we constructed the sequence.
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> Next, we cannot have $c < r$ either:
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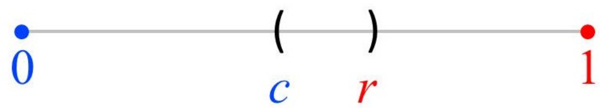
> 
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> —because at every stage of the construction we distinguished a new interval with a cyan left endpoint and a red right endpoint, and the interval (c, r) would have to lie *within* all of those intervals and its

length, a positive number, would have to be *less* than the lengths of all of them. The lengths of those intervals are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, and there is no positive number less than all of those; we have HALVED, AND HAVE NAUGHT. This is a contradiction.

The only possibility left is that c and r are the same, and the claim is settled.

The proof of the claim just given, with its emphasis on halving, is reminiscent of Bolzano's own treatment of the intermediate value theorem [**2**, Section 12]. All that remains is to show that $f(c) = 0$. If $f(c)$ were positive, then c would have a positive AURA that would nonetheless CAPTURE a red number—a contradiction. And if $f(c)$ were negative, then c would have a negative AURA that would nonetheless CAPTURE a cyan number—also a contradiction. So $f(c)$ must indeed be 0.

This takes care of a special case. For the general case, students need to make two observations. First, the endpoints of the interval do not matter, because the HALVE-AND-HAVE-NAUGHT COROLLARY applies for any starting interval length d and because, by propositions already proved here and in the article, the sequence of cyans (resp. reds) must have a perfect ceiling (resp. floor) even if its terms are not rational. Second, the desired value p does not have to be 0 but can be any number between $f(a)$ and $f(b)$, because cyan and red can simply mean "more than p" and "less than p," or vice versa. This proves the IVT.

References

1. M. J. Mansfield, *Introduction to Real Analysis*, Prindle, Weber & Schmidt, Incorporated, Boston, 1969.
2. S. B. Russ, A translation of Bolzano's paper on the intermediate value theorem, *Historia Mathematica*, (2) **7** (May 1980) 156–185.
3. S. G. Simpson, *Subsystems of Second Order Arithmetic*, Springer, Berlin, 1999.